Mathematics Methods
Unit 3 & 4

Random variables

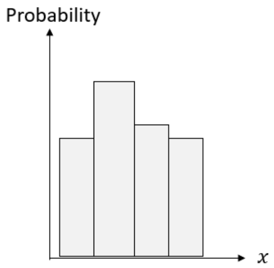
1. Random variables

Definition: A variable whose possible values are outcomes of a random phenomenon

(a) Discrete random variable

Definition: a variable whose values (countable values) are obtained by counting

Example:
- Number of cats in SPCA
- Number of marbles in a jar
- Number of staffs in an office



(i) Probability distribution

Definition: is a list of all of the possible outcomes of a random variable along with their corresponding probability.

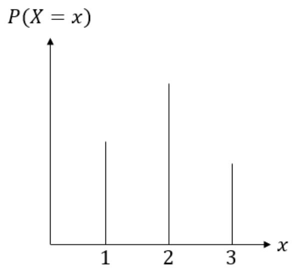
Probability distribution properties (characteristics):
- Probabilities for each value of X lies in the interval of $0 \leq P(X = x) \leq 1$
- Sum (total) of the probabilities is 1

(x can be negative but $P(X = x)$ cannot be negative)

Probability distribution (discrete random variable) can be given by:
- Table form

x							
$P(X = x)$							

- Graphical form



- Function form

$$P(x) = P(X = x)$$

Probability of a discrete random variable by choosing with replacement

$P(X = x) = {}^{n}C_r (p)^r (1-p)^{n-r}$

Example:
A bag of chips contains 4 red and 2 blue chips. Three chips are drawn randomly without replacement. Draw a probability distribution for X: number of blue chips drawn.

$P(red) = \frac{4}{6} = \frac{2}{3}$ $\quad P(blue) = \frac{2}{6} = \frac{1}{3}$

$f(x) = {}^{3}C_x (\frac{1}{3})^x (\frac{2}{3})^{3-x}$ for $x = 0,1,2, \dots$

X	0	1	2
$P(X = x)$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{9}$

or

X	0 (r,r,r)	1 $(r,r,b),(b,r,r)(r,b,r)$	2 $(r,b,b),(b,b,r)(b,r,b)$
$P(X = x)$	$\frac{4}{6} \times \frac{4}{6} \times \frac{4}{6} = \frac{8}{27}$	$3(\frac{4}{6} \times \frac{4}{6} \times \frac{2}{6}) = \frac{4}{9}$	$3(\frac{4}{6} \times \frac{2}{6} \times \frac{2}{6}) = \frac{2}{9}$

Probability of a discrete random variable by choosing without replacement

$$P(X = x) = \frac{{}^{n_1}C_r \, {}^{n_2}C_{n-r}}{{}^{n_1+n_2}C_n}$$

Example:
A bag of marbles contains 4 yellow-green and 3 blue-magenta marbles. Three marbles are drawn randomly without replacement. Draw a probability distribution for X: number of yellow-green marble.

$P(yellow - green) = \frac{4}{7}$ $\quad P(blue - magenta) = \frac{3}{7}$

$$f(x) = \frac{{}^{4}C_x \, {}^{3}C_{3-x}}{{}^{4+3}C_3}$$

X	0	1	2	3
$P(X = x)$	$\frac{1}{35}$	$\frac{12}{35}$	$\frac{18}{35}$	$\frac{4}{35}$

(ii) Interval of values

Interval of values which the variable take is considered in discrete random variable.

Examples:

$P(X = 1)$: the probability that X is equivalent to 1
$P(X > 1)$: the probability that X is more than 1
$P(X \geq 1)$: the probability that X is 1 and above
$P(X < 1)$: the probability that X is less than 1
$P(X \leq 1)$: the probability that X is 1 and below
$P(1 < X < 5)$: the probability that X is between 1 and 5
$P(1 \leq X \leq 5)$: the probability that X is at least 1 and no more than 5
$P(1 < X \leq 5)$: the probability that X is more than 1 and no more than 5
$P(1 \leq X < 5)$: the probability that X is at least 1 and less than 5
$P(X \leq 5|X \geq 1)$: the probability that there is 5 and below given that there is at least 1..

(iii) Expected value/ mean of *X*

Theoretical method

$$E(X) = \sum x \times P(X = x)$$

Example:

Calculate the expected value of X from the given table below.

x	0	1	2	3
$P(X = x)$	$\frac{3}{20}$	$\frac{7}{20}$	$\frac{5}{20}$	$\frac{5}{20}$

Expected value,

$$E(X) = \sum x \times P(X = x)$$
$$= 0\left(\frac{3}{20}\right) + 1\left(\frac{7}{20}\right) + 2\left(\frac{5}{20}\right) + 3\left(\frac{5}{20}\right)$$
$$= 0 + \frac{7}{20} + \frac{10}{20} + \frac{15}{20}$$
$$= 1.6$$

Experimental method

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Example:

Calculate the mean score below.

x	0	1	2	3
Frequency	3	7	5	5

$$\bar{x} = \frac{\sum fx}{\sum f}$$
$$= \frac{3(0) + 7(1) + 5(2) + 5(3)}{3 + 7 + 5 + 5}$$
$$= 1.6$$

Application problem of expected value/ mean

Example:
At a school carnival, Linda is in charge of operating a game stall. The table below shows the prize offered per attempt and its respective probability.

Prize	\$20	\$5	\$1
Probability	0.001	0.01	0.5

Find

- The expected profit per game for Linda if each game cost \$1.

Let X: Profit for Linda

x	-19 $(1-20)$	-4 $(1-5)$	0 $(1-1)$	1 $(1-0)$
$P(X=x)$	0.001	0.01	0.5	0.489 $(1-0.001-0.01-0.5)$

$$E(x) = -19\,(0.001) + -4(0.01) + 0(0.5) + 1(0.489)$$
$$= \$0.43$$

- A customer played the game and paid \$3, find his expected profit/loss.

Let X: Profit for customer

x	17 $(20-3)$	2 $(5-3)$	-2 $(1-3)$	-3 $(0-3)$
$P(X=x)$	0.001	0.01	0.5	0.489

$$E(x) = 17(0.001) + 2(0.01) + -2(0.5) + -3(0.489)$$
$$= -\$2.43$$

- How much Linda charge per game if she made a profit of \$200 from 150 games?

Let X: Profit for Linda, k for price per game

x	$k-20$	$k-5$	$k-1$	k
$P(X=x)$	0.001	0.01	0.5	0.489

$$E(x) = 200 \div 150$$
$$= 1.33 \text{ (profit per game)}$$

$$E(x) = (k-20)(0.001) + (k-5)(0.01) + (k-1)(0.5) + k(0.489)$$
$$1.33 = (k-20)(0.001) + (k-5)(0.01) + (k-1)(0.5) + k(0.489)$$
$$1.33 = 0.001k - 0.02 + 0.01k - 0.05 + 0.5k - 0.5 + 0.489k$$
$$k = 1.9$$

(iv) Variance

Formulas:

$$Var(X) = \sum (x-\mu)^2 \times P(X=x)$$

Or

$$Var(X) = E(X^2) - [E(X)]^2$$

Example:

x	0	1	2	3
$P(X=x)$	0.4	0.3	0.1	0.2

Calculate the variance from the probability distribution above.

$$\begin{aligned} E(X) &= \sum x \times P(X=x) \\ &= 0(0.4) + 1(0.3) + 2(0.1) + 3(0.2) \\ &= 0 + 0.3 + 0.2 + 0.6 \\ &= 1.1 \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= 0^2(0.4) + 1^2(0.3) + 2^2(0.1) + 3^2(0.2) - 1.1^2 \\ &= 2.5 - 1.21 \\ &= 1.29 \end{aligned}$$

Or

$$Var(X) = \sum (x-\mu)^2 \times P(X=x)$$

x	0	1	2	3	Total
$(x-\mu)^2$	1.21	0.01	0.81	3.61	
$P(X=x)$	0.4	0.3	0.1	0.2	
$(x-\mu)^2 \times P(X=x)$	0.484	0.003	0.081	0.722	1.29

$\therefore Var(X) = 1.29$

(v) Standard deviation

Formula:

$$Std(X) = \sqrt{Var(X)}$$

Example:

x	0	1	2	3
$P(X=x)$	0.4	0.3	0.1	0.2

Calculate the standard deviation from the probability distribution above.

$Var(X) = 1.29$

$$\begin{aligned} Std(X) &= \sqrt{Var(X)} \\ &= \sqrt{1.29} \\ &= 1.136 \end{aligned}$$

(vi) Effect of $\times a$ and $+b$

Expected value,

$$E(aX + b) = aE(X) + b$$

Variance

$$Var(aX + b) = a^2Var(X)$$

Standard deviation

$$\sqrt{Var(aX + b)} = \sqrt{a^2Var(X)}$$
$$= |a|\sqrt{Var(X)}$$

Example 1:
A spinning wheel has four equal sections. Each section states the prize won by a contestant namely \$1, \$2 and \$3 If the pay-out is changed to \$0, \$2 and \$4, find the expected value after the change by first determining the expected value of the original prize to be given and then determine a linear rule $Y = aX + b$ to determine the new expected value.

$$E(X) = \sum x \times P(X = x)$$
$$= 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)$$
$$= \frac{1}{3} + \frac{2}{3} + 1$$
$$= 2$$

X	1	2	3
Y	0	2	4

$Y = aX + b,$

When $X = 1$ and $Y = 0,$
$1 = a(0) + b$
$b = 1$

When $X = 2, Y = 2$ and $b = 1,$
$2 = a(2) + 1$
$1 = 2a$
$a = 0.5$

$\therefore Y = 0.5X + 1$

$$E(Y) = 0.5E(X) + 1$$
$$= 0.5(2) + 1$$
$$= 2$$

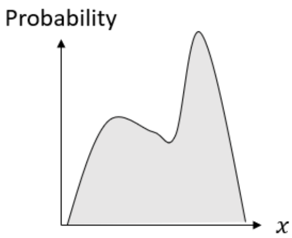
Example 2:
Find $E[(4X - 3)]^2$ given that $E(X) = \frac{1}{2}$ and $E(X^2) = \frac{5}{10}$

$$E[(2X - 1)]^2 = E(4x^2 - 4x + 1)$$
$$= 4\,E(X^2) - 4E(X) + 1$$
$$= 4(\frac{5}{10}) - 4(\frac{1}{2}) + 1$$
$$= 1$$

(b) Continuous random variable

Definition: a variable which takes any values over intervals and whose values (measurable values) are obtained by measuring.

Example:
- Weight of elephants in the national zoo
- Height of green been seedlings after a week
- Diameter of skull



*any graph that has a region enclosed under it

(i) Probability density function

Definition: Probability density function is a function whose value at any given sample in the sample space can be interpreted as providing a relative likelihood that the value of the random variable would equal that sample.

Probability density function (characteristics):
- Sum (total) of the probabilities is 1: $\int_b^a f(x)\,dx = 1$
- $f(x) \geq 0$ for interval $a \leq x \leq b$
- $P(X = k) = 0$ *which is same as* $\int_k^k f(x)\,dx = 0$
- $P(X \leq k) = P(X < k) + P(X = k)$
  $= P(X < k)$

Finding probability density function given a cumulative distribution function

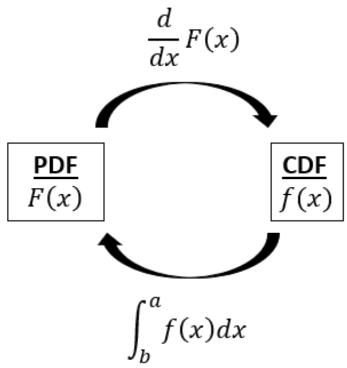
$$f(x) = \frac{d}{dx}F(x)$$

(ii) Cumulative distribution function

Definition: Cumulative distribution function expresses the probability that X does not exceed the value of x.

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

(iii) **PDF and CDF**



Example:
The probability density function f of a continuous random variable T is given by,

$$f(t)\begin{cases}\frac{1}{24}t & 0\le t\le 4\\ \frac{1}{4}-\frac{1}{48}t & 4\le t\le 12\\ 0 & \text{Otherwise}\end{cases}$$

- Find the cumulative distribution function for T.

$$F(t)\begin{cases}0 & t<0\\ \int\frac{1}{24}t\,dt=\frac{t^2}{48} & 0\le t\le 4\\ \int_4^t\frac{1}{4}-\frac{1}{48}t\,dt+\int_0^4\frac{t}{24}dt=\frac{t}{4}-\frac{t^2}{96}-\frac{1}{2} & 4\le t\le 12\\ 1 & t>12\end{cases}$$

- Find $P(3<T<12)$.

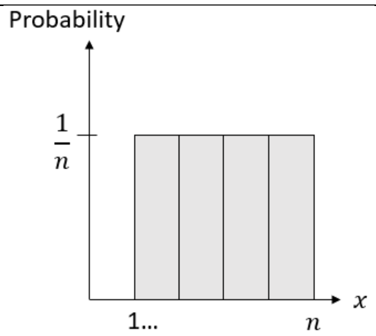
$$P(3<T<12)=F(12)-F(3)$$
$$=\frac{12}{4}-\frac{(12)^2}{96}-\frac{1}{2}-\frac{3^2}{48}$$
$$=\frac{13}{16}$$

or

$$P(3<T<12)=\int_4^{12}\frac{1}{4}-\frac{1}{48}t\,dt+\int_3^4\frac{t}{24}dt$$
$$=\frac{13}{16}$$

	(iv)	**Expected value/ mean of *X*** $$E(X) = \int_a^b x \times f(x)\,dx$$ Example: Random variable X has a probability function of $f(x) = \frac{e^x}{2}$ for $0 \le x \le \ln 3$. Determine the mean for X. $$E(X) = \int_0^{\ln 3} x \times \frac{e^x}{2}\,dx = \frac{3\ln 3 - 2}{2}$$
	(v)	**Variance** $$Var(X) = \int_a^b x^2 \times f(x)\,dx - [E(X)]^2$$ Example: Random variable X has a probability function of $f(x) = 2x$ for $0 \le x \le 1$. Determine the variance for X. $$Var(X) = \int_0^1 x^2 \times 2x\,dx - [\int_0^1 x \times 2x\,dx]^2 = \frac{1}{18}$$
3.	**Uniform distribution**	

(a) Discrete uniform distribution



Discrete uniform variable properties (characteristics):

- n values in the range has equal probability $\frac{1}{n}$
  (the probability of uniformly spaced possible values is equal)

Probability mass function:

$$P(X = x) = \frac{1}{n} \text{ for } x = 1,2,3,4 \ldots., n$$

Notation:

$$X \sim U\{a, b\}$$

	(i)	**Mean/ expected value** Formula: $$E(X) = \frac{n+1}{2}$$ Derivation of formula: $E(X) = \sum x\ P(X = x)$ $= 1\left(\frac{1}{n}\right) + 2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right) + 4\left(\frac{1}{n}\right) + \cdots. + n\left(\frac{1}{n}\right) *$ $= \frac{1+n}{2}$ $* S_n = \frac{n}{2}[2a + (n-1)d]$ $= \frac{n}{2}\left[2\left(\frac{1}{n}\right) + (n-1)\left(\frac{1}{n}\right)\right]$ $= \frac{n}{2}[\frac{2+n-1}{n}]$ $= \frac{n+n^2}{2n}$ $= \frac{1+n}{2}$
		Example: Given that a fair die is rolled and let Z implies that a number appears on the face of the die. Find the mean of Z. $E(X) = \frac{n+1}{2}$ $= \frac{6+1}{2}$ $= 3.5$ Example 2: Given that a discrete uniform variable is given by $U\{1,7\}$ and that $P(X = x) = \frac{1}{7}$. Find the value of the mean. $E(X) = \frac{n+1}{2}$ $= \frac{7+1}{2}$ $= 4$
	(ii)	**Variance** Formula: $$Var(X) = \frac{n^2-1}{12}$$

Derivation of formula:

$$Var(X) = E(X^2) - [E(X)]^2 *$$
$$= \frac{(n+1)(2n+1)}{6} - (\frac{1+n}{2})^2$$
$$= \frac{2n^2+n+2n+1}{6} - (\frac{n^2+2n+1}{4})$$
$$= \frac{4n^2+6n+2-(3n^2+6n+3)}{12}$$
$$= \frac{n^2-1}{12}$$

$$* E(X^2) = \sum x^2 \times P(X=x)$$
$$= 1^2\left(\frac{1}{n}\right) + 2^2\left(\frac{1}{n}\right) + 3^2\left(\frac{1}{n}\right) + 4^2\left(\frac{1}{n}\right) + \cdots . + n^2\left(\frac{1}{n}\right)$$
$$= \frac{1}{n}[1^2+2^2+3^2+4^2]$$
$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]$$
$$= \frac{(n+1)(2n+1)}{6}$$

$$[E(X)]^2 = (\frac{1+n}{2})^2$$

Example 1:

Suppose that a discrete uniform variable is given by $U\{1,7\}$. Find the value of the variance.

$$Var(X) = \frac{n^2-1}{12}$$
$$= \frac{(7)^2-1}{12}$$
$$= 4$$

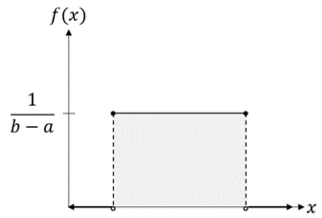
Example 2:

k and h each has a discrete distribution which is uniform for the integers 1,2,3,4, …. n. Show that $Var(k) + Var(h) = \frac{n^2-1}{6}$.

$$Var(k) = Var(h)$$
$$= \frac{n^2-1}{12}$$

$$Var(k) + Var(h) = \frac{n^2-1}{12} + \frac{n^2-1}{12}$$
$$= \frac{2n^2-2}{12}$$
$$= \frac{2(n^2-1)}{12}$$
$$= \frac{n^2-1}{6} \ (shown)$$

(b) Continuous uniform distribution



Probability density function:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Notation:

$$X \sim U(a, b) \text{ if } a < x < b$$
$$X \sim U[a, b] \text{ if } a \le x \le b$$

(i) Mean/ expected value

Formula:

$$E(X) = \frac{a+b}{2}$$

Derivation of formula:

$$E(X) = \int_a^b x\left(\frac{1}{b-a}\right) dx$$
$$= [\frac{x^2}{2(b-a)}]_a^b$$
$$= \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)}$$
$$= \frac{b^2 - a^2}{2(b-a)}$$
$$= \frac{(b+a)(b-a)}{2(b-a)}$$
$$= \frac{a+b}{2}$$

Example 1:
A continuous random variable X is uniformly distributed in the interval $2 \le x \le 10$.
Find $E(X)$.

$$E(X) = \frac{a+b}{2}$$
$$= \frac{10+2}{2}$$
$$= 6$$

		Example 2: The radius of a circle drawn, R can be any value between 5.5cm and 7.5cm. State the mean for R. $E(R) = \frac{a+b}{2}$ $= \frac{5.5+7.5}{2}$ $= 6.5$
	(ii)	**Variance** Formula: $$Var(X) = \frac{(b-a)^2}{12}$$ Derivation of formula: $Var(X) = \int_a^b x^2 \left(\frac{1}{b-a}\right) dx - (\frac{a+b}{2})^2$ $= [\frac{x^3}{3(b-a)}]_a^b - \frac{(a+b)^2}{4}$ $= \frac{b^3}{3(b-a)} - \frac{a^3}{3(b-a)} - \frac{(a+b)^2}{4}$ $= \frac{b^3-a^3}{3(b-a)} - \frac{(a+b)^2}{4}$ $= \frac{(b-a)(b^2+ba+a^2)}{3(b-a)} - \frac{a^2+2ab+b^2}{4}$ $= \frac{(b^2+ba+a^2)}{3} - \frac{a^2+2ab+b^2}{4}$ $= \frac{4(b^2+ba+a^2)}{12} - \frac{3(a^2+2ab+b^2)}{12}$ $= \frac{a^2-2ab+b^2}{12}$ $= \frac{(b-a)^2}{12}$
		Example 1: A continuous random variable X is uniformly distributed in the interval $84 \le x \le 90$. Find $Var(X)$. $Var(X) = \frac{(b-a)^2}{12}$ $= \frac{(90-84)^2}{12}$ $= 3$

		Example 2: The speed of cars on the road, V can be any values in the interval, $30 \leq V \leq 80$. Find the variance of V. $Var(X) = \frac{(80-30)^2}{12}$ $= 208.33$
	(iii)	**Cumulative density function** Formula: $$P(X \leq x) = \frac{x-a}{b-a}$$ Derivation of formula: $P(X \leq x) = \int_a^x \left(\frac{1}{b-a}\right) dx$ $= [\frac{x}{b-a}]_a^x$ $= \frac{x}{b-a} - \frac{a}{b-a}$ $= \frac{x-a}{b-a}$

END